Linear algebra - Practice problems for midterm

1. Compute the following, or state that it is undefined.

(a) $\begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 3 \\ -2 & 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -1 & 4 \\ 5 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & 3 \\ -2 & 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -1 & 4 \end{bmatrix}$

(e) The inverse of $\begin{bmatrix} 1 & 1 & 3 \\ -2 & 0 & 5 \end{bmatrix}$

(f) The inverse of $\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}$

2. Let $A = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 7 & -1 \\ 2 & 9 & -7 \end{bmatrix}$.

(a) Find a basis for the space of solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

(b) Find all solutions of $A\mathbf{x} = \begin{bmatrix} 4 \\ -2 \\ 18 \end{bmatrix}$.

(c) Find the rank of A.

(d) Find the nullspace of A. (For questions like this you can either write all vectors in the nullspace in terms of free variables, or give a basis).

(e) Find a basis for the column space of A.

3 Let $A = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}$ and suppose that the reduced row-echelon form of A is $H = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (you can check this if you want to practice row reduction).

(a) Find a basis for the row space of A.

(b) Find a basis for the column space of A.

(c) Find a basis for the nullspace of A.

(d) What is the rank of A?

(e) Is every linear system $A\mathbf{x} = \mathbf{b}$ consistent? If not, then what condition does $\mathbf{b}$ have to satisfy to make the system consistent?

4. Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be a linear transformation such that $T\left(\begin{bmatrix}2\\3\end{bmatrix}\right) = \begin{bmatrix}1\\0\\-1\end{bmatrix}$ and $T\left(\begin{bmatrix}1\\3\end{bmatrix}\right) = \begin{bmatrix}-1\\0\\4\end{bmatrix}$.

(a) Find $T\left(\begin{bmatrix}3\\6\end{bmatrix}\right)$.

(b) Find the standard matrix representation of T (you can use this to check your answer to part (a)).

(c) Find the rank of T.

(d) Find the kernel of T.

5. Write down the standard matrix representations for the following linear transformations.

(a) $T : \mathbf{R}^3 \to \mathbf{R}^3$ given by reflection in the xz-plane.

(b) $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by reflection in the line $y = x$.

(c) $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by rotation by $180°$ about the origin.

6. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent vectors in $\mathbf{R}^5$.

(a) Write down a basis for $\text{sp}(\mathbf{v}_1, \mathbf{v}_2)$.

(b) Let $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3$ and $\mathbf{w}_3 = \mathbf{v}_1 - \mathbf{v}_3$.

(i) Is $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ a basis for $\text{sp}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$?

(ii) Is $\{\mathbf{w}_1, \mathbf{w}_2\}$ a basis for $\text{sp}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$?

(iii) Show that $2\mathbf{v}_1 + \mathbf{v_2} - \mathbf{v}_3 \in \text{sp}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$.

7. Which of the following are subspaces of $\mathbf{R}^3$? For those which are subspaces, find a basis.

(a) $\{[x, y, z] \quad : \quad xyz = 0\}$.

(b) $\{[t^2, -t^2, 0] \quad : \quad t \in \mathbf{R}\}$.

(c) $\{[x + y, z, 0] \quad : \quad x, y, z \in \mathbf{R}\}$.

8.

(a) Are the vectors $[1, 2, 1], [3, -1, 3], [2, -3, 3]$ linearly independent?

(b) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ be vectors in $\mathbf{R}^3$. Is is possible that they are linearly independent?

(c) Write down 4 vectors in $\mathbf{R}^4$ which are not linearly independent, but such that any 3 of them are linearly independent.